On the product formula on non-compact Grassmannians

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Abstract

We study the absolute continuity of the convolution $\delta^\natural_{e^X}\star\delta^\natural_{e^Y}$ of two orbital measures on the symmetric space $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, $q>p$. We prove sharp conditions on X, $Y\in\mathfrak{a}$ for the existence of the density of the convolution measure. This measure intervenes in the product formula for the spherical functions. We show that the sharp criterion developed for $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$ will also serve for the spaces $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$, $q>p$. We also apply our results to the study of absolute continuity of convolution powers of an orbital measure $\delta^\natural_{e^X}$.

1 Introduction

The spaces $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$ where $q>p$ (which we will assume throughout the paper), are the noncompact duals of real Grassmannians. They are Riemannian symmetric spaces of noncompact type corresponding to root systems of type $\Longrightarrow B_p \Longleftarrow$. The harmonic analysis on these spaces has been intensely developed in recent years ([1, 13, 14, 15]).

We use throughout the paper the usual notations of the harmonic analysis on Riemannian symmetric spaces. The books [9, 10] constitute a standard reference on these topics.

Let X, $Y\in\mathfrak{a}$ and let m_K denote the Haar measure of the group K. We define $\delta^\natural_{e^X}=m_K\star\delta_{e^X}\star m_K$. The question of the absolute continuity of the convolution $\delta^\natural_{e^X}\star\delta^\natural_{e^Y}$ of two K-invariant orbital measures that we address in our paper has important applications in harmonic analysis itself (the product formula for the spherical functions) and in probability theory (random walks, I_0 characterization of Gaussian measures).

The spherical Fourier transform of the measure $\delta^\natural_{e^X}$ is equal to the spherical function $\phi_\lambda(e^X)$, where λ is a complex-valued linear form on $\mathfrak{a}$. Thus the product $\phi_\lambda(e^X)\phi_\lambda(e^Y)$ is the spherical Fourier transform of the convolution $m_{X,Y}=\delta^\natural_{e^X}\star\delta^\natural_{e^Y}$. If we denote by $\mu_{X,Y}$ the projection of the measure $m_{X,Y}$ on $\mathfrak{a}$ via the Cartan decomposition $G=KAK$, then

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y)=\int_{\mathfrak{a}}\phi_\lambda(e^H)\,d\mu_{X,Y}(H).$$

Let δ be the density of the invariant measure on $\mathfrak{a}$ in polar coordinates. The existence of a kernel in the last product formula

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y)=\int_{\mathfrak{a}^+}\phi_\lambda(e^H)\,k(H,X,Y)\,\delta(H)\,dH \tag{1}$$

is equivalent to the absolute continuity of the measure $\mu_{X,Y}$ with respect to the Lebesgue measure on $\mathfrak{a}$ and to the existence of the density of $m_{X,Y}$ on G, with respect to the invariant measure dg. When the formula (1) holds, we say that we have a product formula for X and $Y\in\mathfrak{a}$. Provided that X, $Y\in\mathfrak{a}^+$, the product formula (1) has been shown previously (see [2] in the rank one case,

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[3] in the complex case and [4] in the general case). In [4] we were able to relax these conditions and show that $\mu_{X,Y}$ is absolutely continuous provided one of X or Y is in $\mathfrak{a}^+$ as long as the other is nonzero. The density can however exist in some cases when both X and Y are singular. It is a challenging problem to characterize all such pairs X and Y.

This problem was solved in [7] for symmetric spaces with root system of type A_n. We solve it in this paper for the space $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$: we give a definition of an eligible pair (X,Y) (Definition 3) and next we prove the necessity (Proposition 5) and the sufficiency (Proposition 5 and Theorem 13) of this property for the absolute continuity of $m_{X,Y}$.

$====\Rightarrow$By [3, 4], the density $k(H,X,Y)$ exists if and only if $\mathcal{S}_{X,Y} = a(e^X\,K\,e^Y)$, the support of the measure $\mu_{X,Y}\big|_{\overline{\mathfrak{a}^+}}$, has nonempty interior. Similarly, the density of the measure $m_{X,Y}$ exists if and only if its support Ke^XKe^YK has nonempty interior as seen in [7]. These facts are crucial in the proofs of the results of this paper.

We show in Corollary 15 that the result for the space $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$ also implies the result for the spaces $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$. We conclude the paper with two further applications of our main result. One of them is a characterization of an optimal convolution power l of the measure $\delta^\natural_{e^X}$, which is absolutely continuous for any $X\neq 0$, $X\in\mathfrak{a}$. Theorem 17 solves on non-compact Grassmannians a problem raised by Ragozin in [12]. $\Leftarrow====$

2 Basic properties

We start by reviewing some useful information on the Lie group $\mathbf{SO}_0(p,q)$, its Lie algebra $\mathfrak{so}(p,q)$ and the corresponding root system. Most of this material was given in [15]. For the convenience of the reader, we gather below the properties we will need in the sequel.

In this paper, E_{ij} is a rectangular matrix with 0's everywhere except at at the position (i,j) where it is 1.

Recall that $\mathbf{SO}(p,q)$ is the group of matrices $g\in\mathbf{SL}(p+q,\mathbf{R})$ such that $g^T\,I_{p,q}\,g = I_{p,q}$ where $I_{p,q} = \begin{bmatrix} -I_p & 0_{p\times q}\\ 0_{q\times p} & I_q\end{bmatrix}$. Unless otherwise specified, all 2×2 block decompositions in this paper follow the same pattern.

The group $\mathbf{SO}_0(p,q)$ is the connected component of $\mathbf{SO}(p,q)$ containing the identity. The Lie algebra $\mathfrak{so}(p,q)$ of $\mathbf{SO}_0(p,q)$ consists of the matrices

$$\begin{bmatrix} A & B\\ B^T & D\end{bmatrix}$$

where A and D are skew-symmetric.

A very important element in our investigations is the Cartan decomposition of $\mathfrak{so}(p,q)$ and $\mathbf{SO}(p,q)$. The maximal compact subgroup K is the subgroup of $\mathbf{SO}(p,q)$ consisting of the matrices

$$\begin{bmatrix} A & 0\\ 0 & D\end{bmatrix}$$

of size $(p+q)\times(p+q)$ such that $A\in\mathbf{SO}(p)$ and $D\in\mathbf{SO}(q)$ (hence $K\simeq\mathbf{SO}(p)\times\mathbf{SO}(q)$). If $\mathfrak{k}$

is the Lie algebra of K and $\mathfrak{p}$ is the set of matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

then the Cartan decomposition is given by $\mathfrak{so}(p,q) = \mathfrak{k} \oplus \mathfrak{p}$ with corresponding Cartan involution $\theta(X) = -X^T$.

The Cartan space $\mathfrak{a} \subset \mathfrak{p}$ is the set of matrices

$$H = \begin{bmatrix} 0_{p\times p} & \mathcal{D}_H & 0_{p\times(q-p)} \\ \mathcal{D}_H & 0_{p\times p} & 0_{p\times(q-p)} \\ 0_{(q-p)\times p} & 0_{(q-p)\times p} & 0_{(q-p)\times(q-p)} \end{bmatrix}$$

where $\mathcal{D}_H = \mathrm{diag}[H_1, \ldots, H_p]$. Its canonical basis is given by the matrices

$$A_i := E_{i,p+i} + E_{p+i,i}, 1 \leq i \leq p.$$

The restricted roots and associated root vectors for the Lie algebra $\mathfrak{so}(p,q)$ with respect to $\mathfrak{a}$ are given in Table 1.

root α	multiplicity	root vectors X_α
$\alpha(H) = \pm H_i$ $1 \leq i \leq p$	$q-p$	$X_{ir}^{\pm} = E_{i\,2p+r} + E_{2\,p+r\,i} \pm (E_{p+i\,2p+r} - E_{2p+r\,p+i})$ $r = 1, \ldots, q-p$
$\alpha(H) = \pm(H_i - H_j)$ $1 \leq i,j \leq p,\ i<j$	1	$Y_{ij}^{\pm} = \pm(E_{ij} - E_{ji} + E_{p+i\,p+j} - E_{p+j\,p+i}) + E_{i\,p+j} + E_{p+j\,i}$ $+ E_{j\,p+i} + E_{p+i\,j}$
$\alpha(H) = \pm(H_i + H_j)$ $1 \leq i,j \leq p,\ i<j$	1	$Z_{ij}^{\pm} = \pm(E_{ij} - E_{ji} - E_{p+i\,p+j} + E_{p+j\,p+i}) - (E_{i\,p+j} + E_{p+j\,i})$ $+ E_{j\,p+i} + E_{p+i\,j}$

Table 1: Restricted roots and associated root vectors

The positive roots can be chosen as $\alpha(H) = H_i \pm H_j$, $1 \leq i < j \leq p$ and $\alpha(H) = H_i$, $i = 1, \ldots, p$. We therefore have the positive Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a}\colon\ H_1 > H_2 > \cdots > H_p > 0\}.$$

The simple roots are given by $\alpha_i(H) = H_i - H_{i+1}$, $i = 1, \ldots, p-1$ and $\alpha_p(H) = H_p$.

The action of the Weyl group. The elements of the Weyl group W act as permutations of the diagonal entries of $\mathcal{D}_X$ with eventual sign changes of any number of these entries.

The Lie algebra $\mathfrak{k}$ is generated by the vectors $X_\alpha + \theta X_\alpha$. We will use the notation

$$k_{X_\alpha}^t = e^{t(X_\alpha + \theta X_\alpha)}.$$

The linear space $\mathfrak{p}$ has a basis formed by $A_i \in \mathfrak{a}$, $1 \leq i \leq p$ and by the symmetric matrices $X_\alpha^s := \frac{1}{2}(X_\alpha - \theta X_\alpha)$ which have the following form

$$\begin{aligned} X_{ir} &:= E_{i,2p+r} + E_{2p+r,i}; 1 \leq i \leq p, \qquad 1 \leq r \leq q-p; \\ Y_{ij} &:= E_{i,p+j} + E_{j,p+i} + E_{p+j,i} + E_{p+i,j}, \quad 1 \leq i < j \leq p; \\ Z_{ij} &:= E_{i,p+j} - E_{j,p+i} + E_{p+j,i} - E_{p+i,j}, \quad 1 \leq i < j \leq p. \end{aligned}$$

If we followed the notation of [7], we should write $(X_{ir}^{+})^s$, etc. but we simplify the notation to X_{ir}, Y_{ij} and Z_{ij}. If we write a matrix from the space $\mathfrak{p}$ in the form

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_1 & B_2 \\ B_1^T & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

where B_1 is a square $p \times p$ matrix and B_2 is a $p \times (q-p)$ matrix, then the matrices

$$\begin{bmatrix} 0 & B_1 & 0 \\ B_1^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are generated by the vectors A_i (for the diagonal entries of B_1 and B_1^T), Y_{ij} and Z_{ij} (for the non-diagonal entries), whereas the matrices

$$\begin{bmatrix} 0 & 0 & B_2 \\ 0 & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

are spanned by the vectors X_{ir}.

We now recall the useful matrix $S \in \mathbf{SO}(p+q)$ which allows us to diagonalize simultaneously all the elements of $\mathfrak{a}$. Let

$$S = \begin{bmatrix} \frac{\sqrt{2}}{2} I_p & 0_{p\times(q-p)} & \frac{\sqrt{2}}{2} J_p \\ \frac{\sqrt{2}}{2} I_p & 0_{p\times(q-p)} & -\frac{\sqrt{2}}{2} J_p \\ 0_{(q-p)\times p} & I_{q-p} & 0_{(q-p)\times p} \end{bmatrix}$$

where $J_p = (\delta_{i,p+1-i})$ is a matrix of size $p \times p$. If $H = \begin{bmatrix} 0 & \mathcal{D}_H & 0 \\ \mathcal{D}_H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with $\mathcal{D}_H = \mathrm{diag}[H_1, \ldots, H_p]$ then $S^T\, H\, S = \mathrm{diag}[H_1, \ldots, H_p, \overbrace{0, \ldots, 0}^{q-p}, -H_p, \ldots, -H_1]$.

The "group" version of this result is as follows:

$$S^T\, e^H\, S = \mathrm{diag}[e^{H_1}, \ldots, e^{H_p}, \overbrace{1, \ldots, 1}^{q-p}, e^{-H_p}, \ldots, e^{-H_1}].$$

Remark 1 *The Cartan projection $a(g)$ on the group $\mathbf{SO}_0(p,q)$, defined as usual by*

$$g = k_1 e^{a(g)} k_2, \quad a(g) \in \overline{\mathfrak{a}^+}, k_1,\ k_2 \in K$$

is related to the singular values of $g \in \mathbf{SO}(p,q)$ in the following way. Recall that the singular values of g are defined as the non-negative square roots of the eigenvalues of $g^T g$. Let us write $H = a(g)$. We have

$$g^T\, g = k_2^T\, e^{2\,H}\, k_2 = (k_2^T\, S)\,(S^T\, e^{2\,H}\, S)\,(S^T\, k_2)$$

where $S^T e^{2H} S$ *is a diagonal matrix with nonzero entries* $e^{2H_1}, \dots, e^{2H_p}, \overbrace{1,\dots,1}^{q-p}, e^{-2H_p}, \dots, e^{-2H_1}$. *Let us write* $a_j = e^{H_j}$. *Thus the set of* $p+q$ *singular values of* g *contains the value 1 repeated* $q-p$ *times and the* $2p$ *values* $a_1, \dots, a_p, a_1^{-1}, \dots, a_p^{-1}$.

Hence, in order to determine $a(g)$, *we can compute the* $p+q$ *singular values of* $g^T g$ *and omit* $q-p$ *values 1 always appearing among them. The* $2p$ *remaining singular values may be ordered* $a_1 \geq \dots \geq a_p \geq a_p^{-1} \geq \dots \geq a_1^{-1}$ *with* $a_1 \geq \dots \geq a_p \geq 1$. *Then*

$$a(g) = \begin{bmatrix} 0 & \mathcal{D}_{a(g)} & 0 \\ \mathcal{D}_{a(g)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad \mathcal{D}_{a(g)} = \mathrm{diag}[\log a_1, \dots, \log a_p]$$

Summarizing, if for $g \in \mathbf{SO}(p,q) \subset \mathbf{SL}(p+q)$ *the* $\mathbf{SL}(p+q)$*-Cartan decomposition writes* $g = k_1 e^{\tilde{a}(g)} k_2$, $k_1, k_2 \in \mathbf{SO}(p+q)$, *then* $\mathcal{D}_{a(g)} = \pi_p(\tilde{a}(g))$, *where* π_p *denotes the projection* $\pi_p(\mathrm{diag}[h_1, \dots, h_{p+q}]) = \mathrm{diag}[h_1, \dots, h_p]$.

Singular elements of $\mathfrak{a}$. In what follows, we will consider singular elements X, $Y \in \partial\mathfrak{a}^+$. As in [7], we need to control the irregularity of X and Y, i.e. consider the simple positive roots annihilating X and Y. A special attention must be paid to the last simple root α_p, different from the roots α_i, $i = 1, \dots, p-1$, that generate a root subsystem of type A_{p-1}. We introduce the following definition of the configuration of $X \in \overline{\mathfrak{a}^+}$.

Definition 2 *Let* $X \in \overline{\mathfrak{a}^+}$. *There exist nonnegative integers* $s_1 \geq 1, \dots, s_r \geq 1$, $u \geq 0$ *such that*

$$\mathcal{D}_X = \mathrm{diag}[\,\overbrace{x_1, \dots, x_1}^{s_1}, \overbrace{x_2, \dots, x_2}^{s_2}, \dots, \overbrace{x_r, \dots, x_r}^{s_r}, \overbrace{0, \dots, 0}^{u}\,]$$

with $x_1 > x_2 > \dots > x_r > 0$ *and* $\sum s_i + u = p$. *We say that* $[s_1, \dots, s_r; u]$ *is the* **configuration** *of* X. *Writing* $\mathbf{s} = (s_1, \dots, s_r)$, *we will shorten the notation of the configuration of* X *to* $[\mathbf{s}; u]$. *We will also write* $X = X[\mathbf{s}; u]$.

Note that $X = 0$ is equivalent to $u = p$ and has configuration $[0; p]$. A regular $X \in \mathfrak{a}^+$ has the configuration $[\mathbf{1}; 0] = [1^p; 0]$. We extend naturally the definition of configuration to any $X \in \mathfrak{a}$, whose configuration is defined as that of the projection $\pi(X)$ of X on $\overline{\mathfrak{a}^+}$.

In what follows, we will write $\max \mathbf{s} = \max_i s_i$ and $\max(\mathbf{s}, u) = \max(\max \mathbf{s}, u)$. We will show that in the case of the symmetric spaces $\mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $q > p$, the criterion for the existence of the density of the convolution $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ is given by the following definition of an eligible pair X and Y:

Definition 3 *Let* $X = X[\mathbf{s}; u]$ *and* $Y = Y[\mathbf{t}; v]$ *be two elements of* $\mathfrak{a}$. *We say that* X *and* Y *are eligible if*

$$\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p.$$

Observe that if X and Y are eligible, then $X \neq 0$ and $Y \neq 0$.

3 Necessity of the eligibility condition

In the proof of the necessity of the eligibility condition we will use the result stated in [8, Step 1, page 1767]:

Lemma 4 *Let* $U = \mathrm{diag}([\overbrace{u_0, \ldots, u_0}^{r}, u_1, \ldots, u_{N-r}]$ *and* $V = \mathrm{diag}([\overbrace{v_0, \ldots, v_0}^{N-s}, v_1, \ldots, v_s]$. *where* $s+1 \leq r < N$, $s \geq 1$, *and the* u_i*'s and* v_j*'s are arbitrary. Then each element of* $\tilde{a}(e^U\,\mathbf{SU}(N, \mathbf{F})\,e^V)$ *has at least* $r - s$ *entries equal to* $u_0 + v_0$.

We will use Lemma 4 with $N = p + q$ in the proofs of Proposition 5 and Theorem 17.

Proposition 5 *If* $X[\mathbf{s}; u]$ *and* $Y[\mathbf{t}, v]$ *are not eligible then the measure* $\mu_{X,Y}$ *is not absolutely continuous with respect to the Lebesgue measure on* $\mathfrak{a}$.

Proof: Suppose $\max(\mathbf{s}, 2\,u) + \max(\mathbf{t}, 2\,v) > 2\,p$ and consider the matrices $a(e^X\,k\,e^Y)$, $k \in \mathbf{SO}(p) \times \mathbf{SO}(q)$. Applying Remark 1, the diagonal $p \times p$ matrix $\mathcal{D}_{a(e^X\,k\,e^Y)}$ contains the p biggest diagonal terms of the matrix

$$\tilde{a}(e^X k e^Y) = \tilde{a}(\overbrace{(S^T\,e^X\,S)}^{e^{S^T\,X\,S}}\,\overbrace{(S^T\,k\,S)}^{\in\mathbf{SO}(p+q)}\,\overbrace{(S^T\,e^Y\,S)}^{e^{S^T\,Y\,S}})$$

If $u + v > p$ then there are $r - s = r + (N - s) - N = (2\,u + q - p) + (2\,v + q - p) - (p + q) = 2\,(u + v - p) + (q - p)$ repetitions of $0 + 0 = 0$ in coefficients of $\tilde{a}(e^X k e^Y)$. Therefore 0 occurs at least $u + v - p > 0$ times as a diagonal entry of $\mathcal{D}_H$ for every $H \in a(e^X\,K\,e^Y)$ which implies that $a(e^X\,K\,e^Y)$ has empty interior.

If $2\,u + \max(\mathbf{t}) > 2\,p$ denote $t = \max(\mathbf{t})$. Let $Y_i \neq 0$ be repeated t times in $\mathcal{D}_Y$. Then there are $r - s = r + (N - s) - N = (2\,u + q - p) + t - (p + q) = 2\,u + t - 2p$ repetitions of $Y_i + 0$ in coefficients of $\tilde{a}(e^X k e^Y)$. Therefore Y_i occurs at least $2\,u + t - 2p > 0$ times as a diagonal entry of $\mathcal{D}_H$ for every $H \in a(e^X\,K\,e^Y)$ which implies that $a(e^X\,K\,e^Y)$ has empty interior. ∎

4 Sufficiency of the eligibility condition

We use basic ideas and some results and notations of [7, Section 3].

Proposition 6 *(i) The density of the measure* $m_{X,Y}$ *exists if and only if its support* Ke^XKe^YK *has nonempty interior.*
(ii) Consider the analytic map $T\colon K \times K \times K \to \mathbf{SO}_0(p, q)$ *defined by*

$$T(k_1, k_2, k_3) = k_1\,e^X\,k_2\,e^Y\,k_3.$$

If the derivative of T *is surjective for some choice of* $\mathbf{k} = (k_1, k_2, k_3)$, *then the set* $T(K \times K \times K) = Ke^XKe^YK$ *contains an open set.*

Proof: Part (i) follows from arguments explained in [4] in the case of the support of the measure $\mu_{X,Y}$, equal to $a(e^X\,Ke^Y)$. Part (ii) is justified for example in [10, p. 479]. ∎

Proposition 7 *Let $U_Z = \mathfrak{k} + Ad(Z)\mathfrak{k}$. If there exists $k \in K$ such that*

$$U_{-X} + Ad(k)U_Y = \mathfrak{g} \tag{2}$$

then the measure $m_{X,Y}$ is absolutely continuous.

Proof: We want to show that the derivative of T is surjective for some choice of $\mathbf{k} = (k_1, k_2, k_3)$.

Let $A, B, C \in \mathfrak{k}$. The derivative of T at $\mathbf{k}$ in the direction of (A, B, C) equals

$$\begin{aligned} dT_{\mathbf{k}}(A, B, C) &= \frac{d}{dt}\Big|_{t=0} e^{tA} k_1\, e^X\, e^{tB} k_2\, e^Y\, e^{tC} k_3 \\ &= A\, k_1\, e^X\, k_2\, e^Y\, k_3 + k_1\, e^X\, B\, k_2\, e^Y\, k_3 + k_1\, e^X\, k_2\, e^Y\, C\, k_3 \end{aligned} \tag{3}$$

We now transform the space of all matrices of the form (3) without modifying its dimension:

$$\begin{aligned} &\dim\{A\, k_1\, e^X\, k_2\, e^Y\, k_3 + k_1\, e^X\, B\, k_2\, e^Y\, k_3 + k_1\, e^X\, k_2\, e^Y\, C\, k_3 \colon A, B, C \in \mathfrak{k}\} \\ &= \dim\{k_1^{-1}\, A\, k_1\, e^X\, k_2\, e^Y + e^X\, B\, k_2\, e^Y + e^X\, k_2\, e^Y\, C \colon A, B, C \in \mathfrak{k}\} \\ &= \dim\{A\, e^X\, k_2\, e^Y + e^X\, B\, k_2\, e^Y + e^X\, k_2\, e^Y\, C \colon A, B, C \in \mathfrak{k}\} \\ &= \dim\{e^{-X}\, A\, e^X + B + k_2\, e^Y\, C\, e^{-Y} k_2^{-1} \colon A, B, C \in \mathfrak{k}\} \end{aligned}$$

The space in the last line equals $\mathfrak{k} + \mathrm{Ad}(e^{-X})(\mathfrak{k}) + \mathrm{Ad}(k_2)\left(\mathrm{Ad}(e^Y)(\mathfrak{k})\right) = U_{-X} + \mathrm{Ad}(k_2)U_Y$. ∎

In order to apply the condition (2), we will consider convenient root vectors and their symmetrizations. For $Z \in \mathfrak{a}$, we define the space

$$V_Z = \mathrm{span}\{X_\alpha^s \mid \alpha(Z) \neq 0\},$$

where $X_\alpha^s = X_\alpha - \theta X_\alpha$. Note that this space would be called V_Z^S in the notation of [7].

Lemma 8 *Let $Z \in \mathfrak{a}$. The vector space $U_Z = \mathfrak{k} + Ad(e^Z)(\mathfrak{k})$ contains the root vectors X_α for which $\alpha(Z) \neq 0$. $\Longrightarrow$Consequently, $V_Z = V_{-Z} \subset U_{\pm Z}$.$\Longleftarrow$*

Proof: $\Longrightarrow$Suppose α is a root such that $\alpha(Z) \neq 0$.$\Longleftarrow$Note that $[Z, X_\alpha] = \alpha(Z)\, X_\alpha$ and $[Z, \theta(X_\alpha)] = -\alpha(Z)\, \theta(X_\alpha)$. Let $U = X_\alpha + \theta(X_\alpha) \in \mathfrak{k}$. Now,

$$\begin{aligned} \mathrm{Ad}(e^Z)\, U = e^{\mathrm{ad}\, Z}\, (X_\alpha + \theta(X_\alpha)) &= \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\, Z)^k}{k!}\, (X_\alpha + \theta(X_\alpha)) \\ &= \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\, Z)^k}{k!}\, X_\alpha + \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\, Z)^k}{k!}\, \theta(X_\alpha) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha(Z))^k}{k!}\, X_\alpha + \sum_{k=0}^{\infty} \frac{(-1)^k\, (\alpha(Z))^k}{k!}\, \theta(X_\alpha) \\ &= e^{\alpha(Z)}\, X_\alpha + e^{-\alpha(Z)}\, \theta(X_\alpha). \end{aligned}$$

Therefore $X_\alpha = (e^{\alpha(Z)} - e^{-\alpha(Z)})^{-1} \left(-e^{-\alpha(Z)}\, U + \mathrm{Ad}(e^Z)\, U\right) \in \mathfrak{k} + \mathrm{Ad}(e^Z)(\mathfrak{k}) = U_Z$. The vector θX_α is a root vector for the root $-\alpha$, so we also have $\theta X_\alpha \in U_Z$. ∎

$\Longrightarrow$

Proposition 9 *If there exists $k \in K$ such that*

$$V_X + Ad(k)\, V_Y = \mathfrak{p} \tag{4}$$

then the measure $m_{X,Y}$ is absolutely continuous.

Proof: We want to prove formula (2). By Lemma 8, we know that $V_X = V_{-X} \subset U_{-X}$ and $V_Y \subset U_Y$. As $\mathfrak{k} \subset U_X$, we see that the formula (4) implies (2). ∎

$\Longleftarrow====$

Later in this section,(Theorem 13), we will show that the hypotheses of the last Proposition are always satisfied for X and Y eligible. For technical reasons, in order to make an induction proof work, we will show more, i.e. that a "better" matrix $k \in K$ exists such that the formula (4) holds. The meaning of a "better" k will be similar to the notion of a total matrix given in Definition 10. Here is a definition and a lemma about total matrices in K. The reasonning of the proof of this lemma will be used in a more general setting in Steps 2 and 3 of the proof of Theorem 13.

Definition 10 *We say that a square $n \times n$ matrix k is total if by removing any $r < n$ rows and r columns of k we always obtain a nonsingular matrix.*

Note that this definition of totality is more restrictive than in [7, Definition 3.7].

Lemma 11 *The set of matrices in $\mathbf{SO}(n)$ which are total is dense and open in $\mathbf{SO}(n)$.*

Proof: Consider first the set $M_{I,J} = M_{\{i_1,\ldots,i_j\},\{j_1,\ldots,j_r\}} \subset \mathbf{SO}(n)$ of orthogonal matrices which remain nonsingular once the rows of indices $i_1, \ldots, i_j$ and the columns of indices $j_1, \ldots, j_r$ are removed. To see that such matrices exist, take the identity matrix (whose determinant is 1 if we remove, say, the first r rows and columns). By taking convenient permutations of the rows and columns of the identity matrix, we obtain an element of $M_{I,J}$. Given that $\mathbf{SO}(n) \backslash M_{I,J}$ corresponds to the set of zeros of a certain determinant function, it must be closed and nowhere dense in $\mathbf{SO}(n)$.

To conclude, it suffices to notice that the set of total matrices in $\mathbf{SO}(n)$ is the finite intersection of all the sets $M_{I,J}$. ∎

In the proof of the main Theorem 13 we will need the following technical lemma.

Lemma 12 *(i) For the root vectors X_{ir}^+, Z_{ij}^+, Y_{ij}^+, we have*

$$\begin{aligned}
Ad(e^{t\,(X_{ir}^+ + \theta\, X_{ir}^+)})(X_{ir}) &= \cos(2\,t)\, X_{ir} + 2\, \sin(2\,t)\, A_i, \\
Ad(e^{t\,(Y_{ij}^+ + \theta\, Y_{ij}^+)})(Y_{ij}) &= \cos(4\,t)\, Y_{ij} + 2\, \sin(4\,t)\, (A_i - A_j), \\
Ad(e^{t\,(Z_{ij+} + \theta\, Z_{ij}^+)})(Z_{ij}) &= \cos(4\,t)\, Z_{ij} + 2\, \sin(4\,t)\, (A_i + A_j).
\end{aligned}$$

(ii) The functions $Ad(e^{t\,(X_{ir}^+ + \theta\, X_{ir}^+)})$, $Ad(e^{t\,(Y_{ij}^+ + \theta\, Y_{ij}^+)})$ and $Ad(e^{t\,(Z_{ij}^+ + \theta\, Z_{ij}^+)})$ applied to the other symmetrized root vectors do not produce any components in $\mathfrak{a}$.

Proof: It is just a matter of carefully evaluating

$$\mathrm{Ad}(e^{t\,(Z+\theta\,Z)})(W) = e^{t\;\mathrm{ad}(Z+\theta\,Z)}(W) = \sum_{k=0}^{\infty} (\mathrm{ad}(Z+\theta\,Z))^k(W)\,\frac{t^k}{k!}.$$

For (ii), use the well known properties of the root system: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$ and $[X_\alpha, \theta X_\alpha] \in \mathfrak{a}$. ∎

By Proposition 9, in order to justify the sufficiency of the eligibility condition, it is enough to prove the following theorem. This is the main result of this section.

Theorem 13 *Let $G = \mathbf{SO}_0(p,q)$ and let $X,\, Y \in \mathfrak{a}$. If X and Y are eligible then there exists a matrix $k \in K$ such that*

$$V_X + Ad(k)\,V_Y = \mathfrak{p}. \tag{5}$$

Proof: We will assume that $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$. Observe that the spaces V_X and V_Y depend on the Weyl chambers where X and Y belong. However, see [7, Lemma 3.3 and Reduction 1, p. 759], the property (5) is equivalent to $V_{w_1 X} + \mathrm{Ad}(k')\,V_{w_2 Y} = \mathfrak{p}$ for any $w_1, w_2 \in W$ and a convenient $k' \in K$. Throughout the proof we will assume that the diagonal entries of $\mathcal{D}_X$ and $\mathcal{D}_Y$ are non-negative and we will arrange (permute) them conveniently.

To lighten the notation, for a matrix c of size $p \times q$, we will consider the $(p+q)\times(p+q)$ symmetric matrix

$$c^s = \begin{bmatrix} 0 & c \\ c^T & 0 \end{bmatrix} \in \mathfrak{p}.$$

The proof will be organized in the following way:

1. Proof for $q = p+1$ using induction on p
   (a) Proof for $p = 2$ and $q = 3$
   (b) Proof of the induction step
      i. Proof in the case $u > 0$ or $v > 0$
      ii. Proof in the case $X[p;0], Y[p;0]$
2. Proof that the case (p,q) implies the case $(p, q+1)$.

1. Proof for $q = p+1$ using induction on p

(a) Proof for $p = 2$ and $q = 3$

This corresponds to the space $\mathbf{SO}_0(2,3)$. Only two configurations $[2;0]$ and $[1;1]$ may be realized by singular non-zero X and Y. When $Z \in \overline{\mathfrak{a}}^+$, we have $\mathcal{D}_{Z[1;1]} = \mathrm{diag}[z,0]$, and $\mathcal{D}_{Z[2;0]} = \mathrm{diag}[z,z]$, $z \neq 0$. It is easy to check that in all 3 possible cases:

$$(i)\ X[2;0], Y[2;0] \quad (ii) X[2;0], Y[1;1] \text{ or} X[1;1], Y[2;0] \quad (iii) X[1;1], Y[1;1],$$

X and Y are eligible. Note that

$$\begin{aligned}
\mathfrak{p} &= \left\{ \begin{bmatrix} h_1 & a & b \\ c & h_2 & d \end{bmatrix}^s : h_1, h_2, a, b, c, d \in \mathbf{R} \right\}, \\
V_{Z[2;0]} &= \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}, \\
V_{Z[1;1]} &= \left\{ \begin{bmatrix} 0 & a & c \\ b & 0 & 0 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}.
\end{aligned}$$

If $k_1 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ then

$$\mathrm{Ad}(k_1)\, V_{Z[2;0]} = \left\{ \begin{bmatrix} \sqrt{2}\,a/2 & (a-b+c)/2 & (a+b-c)/2 \\ -\sqrt{2}\,a/2 & (a-b-c)/2 & (a+b+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\},$$

and

$$\mathrm{Ad}(k_1)\, V_{Z[1;1]} = \left\{ \begin{bmatrix} -\sqrt{2}\,b/2 & (a-c)/2 & (a+c)/2 \\ \sqrt{2}\,b/2 & (a-c)/2 & (a+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}$$

If $k_2 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$ then

$$\mathrm{Ad}(k_2)\, V_{Z[1;1]} = \left\{ \begin{bmatrix} -(b+c)/2 & \sqrt{2}\,a/2 & (-b+c)/2 \\ (b-c)/2 & \sqrt{2}\,a/2 & (b+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}.$$

We verify easily that in the cases (i) and (iii) we have $V_X + \mathrm{Ad}(k_1)\, V_Y = \mathfrak{p}$. For $X[2;0]$ and $Y[1;1]$, we can see that $V_X + \mathrm{Ad}(k_2)\, V_Y = \mathfrak{p}$.

(b) Proof of the induction step

(i) Proof in the case $u > 0$ or $v > 0$

We consider the space $\mathbf{SO}_0(p, p+1)/\mathbf{SO}(p) \times \mathbf{SO}(p+1)$ with $p > 2$ and the case when $X[\mathbf{s}; u]$ and $Y[\mathbf{t}; v]$ are such that $u > 0$ or $v > 0$. We assume $u \geq v$. We choose the predecessors in $\mathbf{SO}_0(p-1, p)/\mathbf{SO}(p-1) \times \mathbf{SO}(p)$ in the following way:

$$X' = X'[\mathbf{s}; u-1],\ Y' = Y'[\mathbf{t}'; v]$$

where $\mathbf{t}'$ means that we supress one term from the longest block of size $\max \mathbf{t}$. Note that if $p > 2$ then $\mathbf{t}'$ is not the zero partition (otherwise, $\mathbf{t}$ would have been the partition [1] meaning that $u \geq v = p - 1$ which would make X and Y ineligible).

We arrange X, X', Y, Y' in the following way.
1. The first diagonal entry of $\mathcal{D}_X$ is zero and all the other zeros are at the end. The diagonal entries of $\mathcal{D}_{X'}$ are those of $\mathcal{D}_X$ without the first zero:

$$\mathcal{D}_X = \mathrm{diag}[0, \overbrace{x_1, \ldots, x_{p-u}}^{\neq 0}, \overbrace{0, \ldots, 0}^{u-1}], \quad \mathcal{D}_{X'} = \mathrm{diag}[\overbrace{x_1, \ldots, x_{p-u}}^{\neq 0}, \overbrace{0, \ldots, 0}^{u-1}]$$

2. We put a longest block of size t of equal diagonal entries y_1 of $\mathcal{D}_Y$ in the beginning of Y. The diagonal entries of $\mathcal{D}_{Y'}$ are those of $\mathcal{D}_Y$ with the first entry omitted:

$$\mathcal{D}_Y = \mathrm{diag}[\overbrace{y_1, \ldots, y_1}^{t}, y_2, \ldots, y_s], \quad \mathcal{D}_{Y'} = \mathrm{diag}[\overbrace{y_1, \ldots, y_1}^{t-1}, y_2, \ldots, y_s].$$

It is easy to check that if X, Y are eligible in $\mathbf{SO}_0(p, p+1)$ then X', Y' are eligible in $\mathbf{SO}_0(p-1, p)$.

Step 1. By the induction hypothesis, there exists a matrix $k_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ such that

$$V_{X'} + \mathrm{Ad}(k_0) V_{Y'} = \mathfrak{p}'. \tag{6}$$

We embed $K' = \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ in $\mathbf{SO}(p) \times \mathbf{SO}(p+1)$ in the following way

$$K' = \begin{bmatrix} 1 & & & \\ & \mathbf{SO}(p-1) & & \\ & & 1 & \\ & & & \mathbf{SO}(p) \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(p+1) \end{bmatrix}.$$

Hence, we have (taking the natural embedding of $\mathfrak{p}'$ into $\mathfrak{p}$)

$$V_1 := V_{X'} + \mathrm{Ad}(k_0) V_{Y'} = \mathfrak{p}' = \begin{bmatrix} 0 & B' \\ B'^T & 0 \end{bmatrix} \tag{7}$$

where $B' = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times p} \\ \hline 0_{p\times 1} & B''_{(p-1)\times p} \end{array}\right]$ and B'' is arbitrary (note that $\mathfrak{p}'$ is of dimension $(p-1)p$). We must show that for some $k \in K$, the space $V_X + \mathrm{Ad}(k)\, V_Y = \mathfrak{p}$, i.e. that

(i) $V_X + \mathrm{Ad}(k)\, V_Y$ contains $\mathfrak{p}'$ embedded into $\mathfrak{p}$ as in (7).

(ii) $V_X + \mathrm{Ad}(k)\, V_Y$ contains all the matrices of the form

$$C = \left[\begin{array}{c|c} * & * \cdots * \\ \hline * & \\ \vdots & 0_{(p-1)\times p} \\ * & \end{array}\right]^s.$$

New vectors in V_X and V_Y. In order to prove the induction conclusion, we must now use the elements of V_X and V_Y which do not come from $V_{X'}$ or $V_{Y'}$. They appear by the interaction of, respectively, the first diagonal entry of $\mathcal{D}_X$ with the others of $\mathcal{D}_X$ and the interaction of the first entry of $\mathcal{D}_Y$ with the others of $\mathcal{D}_Y$. We see that the new independent root vectors in V_X and V_Y are respectively

$$N_X = \{Y_{1j}, Z_{1j}, j = 2, \dots, p+1-u\}, \qquad N_Y = \{X_1, Y_{1i}, i = t+1, \dots, p,\ Z_{1j}, j = 2 \dots, p\}$$

where $t = \max \mathbf{t} > 1$ and we wrote X_1 for X_{11}. Note that N_X has $2p - 2u$ elements while N_Y has $2\,p - t$.

Step 2. *We show that there exists $k_0' \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ for which (6) holds, and with the following property:*

The space $V_2 := Ad(k_0')\mathrm{span}(N_Y)$ is of dimension $2p - t$ and its elements can be written in the form

$$\left[\begin{array}{c|cccccc} 0 & \sigma_1 & \dots & \sigma_r & a_1 & \dots & a_{p-r} \\ \hline \tau_1 & & & & & & \\ \vdots & & & & & & \\ \tau_s & & & 0 & & & \\ a_{p-r+1} & & & & & & \\ \vdots & & & & & & \\ a_{2\,p-t} & & & & & & \end{array}\right]^s \tag{8}$$

with $r = [(t-1)/2]$, $s = t-1-r$, $a_i \in \mathbf{R}$ arbitrary, $\sigma_i = \sigma_i(a_1, \dots, a_{2\,p-t})$ and $\tau_j = \tau_j(a_1, \dots, a_{2\,p-t})$, $i \leq r, j \leq s$.
We will not need to write explicitely the functions σ_i and τ_j. Note that $s = r$ if t is odd and $s = r + 1$ if t is even.

To justify Step 2, we write

$$k_0 = \begin{bmatrix} 1 & & & \\ & k_{01} & & \\ & & 1 & \\ & & & k_{02} \end{bmatrix}$$

where $k_{01} \in \mathbf{SO}(p-1)$ and $k_{02} \in \mathbf{SO}(p)$. Let $\alpha_1, \dots, \alpha_{p-1}$ be the columns of the matrix k_{01} and $\beta_1, \dots, \beta_p$ the columns of the matrix k_{02}. A simple block multiplication to compute the action of $\mathrm{Ad}(k_0)$ on the elements of N_Y gives the linearly independent matrices

$$\mathrm{Ad}(k_0)X_1 = \left[\begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array}\right]^s, \quad \mathrm{Ad}(k_0)Y_{1i} = \left[\begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline \alpha_{i-1} & 0 \end{array}\right]^s, \ i = t+1, \dots, p,$$
$$Ad(k_0)Z_{1i} = \left[\begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline -\alpha_{i-1} & 0 \end{array}\right]^s, i = 2, \dots, p. \tag{9}$$

Let us write β_i' for a column β_i from which we have removed the first r entries and α_i' for a column α_i with the first s entries omitted. In order to prove the statement of Step 2, we must show that the matrices obtained by replacing β_i by β_i' and α_i by α_i' in (9) are still linearly independent. This is equivalent to the linear independence of the matrices

$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\alpha_i' & 0 \end{array}\right]^s, \ i = 1, \dots, t-1, \quad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array}\right]^s, \ i = t, \dots, p, \quad \left[\begin{array}{c|c} 0 & 0 \\ \hline \alpha_i' & 0 \end{array}\right]^s \ i = t, \dots, p-1. \tag{10}$$

We will reason in the same way as in Lemma 11.

It is enough to show that there exists at least a choice of matrices k_{01} and k_{02} such that the matrices in (10) are linearly independent. Then, as in Lemma 11, it will follow that such matrices form a dense open subset in $\mathbf{SO}(p-1)\times\mathbf{SO}(p)$. By choosing k'_0 with the matrices in (10) linearly independent and close enough to k_0, the property (6) will be preserved for k'_0.

Pick $k_{01}=I_{p-1}$ which implies that $\alpha'_i=\mathbf{0}$ for $i=1,\ldots,s$ and $\alpha'_i=\mathbf{e}_{i-s}$ for $i>s$. With this choice, (10) becomes

$$\left[\begin{array}{c|c}0&\beta_i'^T\\\hline 0&0\end{array}\right]^s,\ i=1,\ldots,s,\quad \left[\begin{array}{c|c}0&\beta_i'^T\\\hline -\mathbf{e}_{i-s}&0\end{array}\right]^s,\ i=s+1,\ldots,t-1,\quad \left[\begin{array}{c|c}0&\beta_i'^T\\\hline 0&0\end{array}\right]^s,\ i=t,\ldots,p,$$
$$\left[\begin{array}{c|c}0&0\\\hline \mathbf{e}_{i-s}&0\end{array}\right]^s\ i=t,\ldots,p-1$$

which are linearly independent provided that

$$\left[\begin{array}{c|c}0&\beta_i'^T\\\hline 0&0\end{array}\right]^s,\ i=1,\ldots,s,\quad \left[\begin{array}{c|c}0&\beta_i'^T\\\hline 0&0\end{array}\right]^s,\ i=t,\ldots,p$$

are linearly independent. This is the case for a total matrix $k_{02}\in\mathbf{SO}(p)$ or by taking convenient permutations of the rows and columns of the identity matrix I_p.

Step 3. *We show that there exists a proper subset N'_X of N_X such that, if*

$$V_3:=\operatorname{span}(N'_X)+V_{X'}+Ad(k'_0)V_Y=\operatorname{span}(N'_X)+V_1+V_2$$

then $\dim V_3=p\,q-1=\dim\mathfrak{p}-1$ *and V_3 is given by*

$$V_3=\left\{\left[\begin{array}{c|c}0&a_1\ \ a_2\ \ \ldots\ \ a_p\\\hline a_{p+1}&\\ \vdots&\mathfrak{p}'\\ a_{2p-1}&\end{array}\right]^s,\quad a_1,\ldots a_{2p-1}\in\mathbf{R}\right\}. \tag{11}$$

Note that in matrices from the space V_2, there are $r=[(t-1)/2]$ pairs (σ_i,τ_i) plus possibly an extra τ_s if t is even and therefore $s=r+1$. Note also that $t+2\,u\le 2\,p$ implies that $p-u\ge s\ge r$. For $j\le r\le p-u$, each pair $Y_{1j}=\left[\begin{array}{c|c}0&\mathbf{e}_{j-1}^T\\\hline \mathbf{e}_{j-1}&0\end{array}\right]^s$, $Z_{1j}=\left[\begin{array}{c|c}0&\mathbf{e}_{j-1}^T\\\hline -\mathbf{e}_{j-1}&0\end{array}\right]^s$ of elements of N_X allows us to replace σ_j and τ_j by independent variables. If t is odd, all the σ_j's and τ_j's will be taken care off and at least 2 elements of N_X will remain off N'_X. If t is even, all the σ_j and τ_j's, $1\le j\le r$ will be replaced by independent variables and only τ_s will remain. Now, letting the coefficient a_1 "vis-à-vis" the remaining τ_s be equal to 1 and all the other variables a_i equal to 0, either $\tau_s=1$ or -1 or $\tau_s\neq\pm1$. If $\tau_s=1$ then Z_{1s} allows us to introduce the missing independent variable, if $\tau_s=-1$ then adding Y_{1s} to N'_X will do the trick. In the case $\tau_s\neq\pm1$ we choose indifferently between Y_{1s} and Z_{1s}. In all cases the set $N_X\setminus N'_X$ has at least one element.

Step 4. *Let v_1 be the positive root vector corresponding to an element of $N_X\setminus N'_X$. We denote $k_1^t=k_{v_1}^t$. There exists $\epsilon>0$ such that for $t\in(0,\epsilon)$*

$$V_4^t:=\operatorname{Ad}(k_1^t)(\operatorname{span}(N'_X)+V_{X'})+\operatorname{Ad}(k'_0)V_Y=V_3.$$

Observe that v_1 is equal to Z_{1j}^+ or Y_{1j}^+ for one of the remaining Z_{1j} or Y_{1j} that was not used in the preceding step. The space $V_4^t \subset V_4^0$ for all t according to Lemma 12 (ii).

Recall the definition of $k_{X_\alpha}^t = e^{t(X_\alpha+\theta X_\alpha)}$, $t>0$. Let $d(t) = \dim V_4^t$; for $t=0$ we have $k_1^0 = Id$, and $\mathrm{Ad}(k_1^0)(\mathrm{span}(N'_X) + V_{X'}) + \mathrm{Ad}(k'_0)V_Y = V_3$ is of dimension $pq-1$, so $d(0) = pq-1$. The equality $d(t) = d(0)$ is equivalent to non-nullity of an appropriate determinant continuous in t. Thus $d(t) = pq-1$ holds for $t \in (0,\epsilon)$ for some $\epsilon > 0$. As $V_4^t \subset V_4^0$, the statement of Step 4 follows.

Step 5. *Generation of* A_1. By Lemma 12, we have $\mathrm{Ad}(k_1^t)v_1^s = a_t v_1^s + b_t A_1 + c_t A_j$ with $j \neq 1$ and $b_t \neq 0$ for $t \in (0,\epsilon)$ with ϵ small enough. Consequently

$$\mathrm{Ad}(k_1^t)\mathrm{span}(v_1^s) + V_4^t = \mathfrak{p}.$$

Conclusion. We have $\mathfrak{p} = \mathrm{Ad}(k_1^t)(\mathbf{R}v_1^s + \mathrm{span}N'_X + V_{X'}) + \mathrm{Ad}(k'_0)V_Y \subset \mathrm{Ad}(k_1^t)V_X + \mathrm{Ad}(k'_0)V_Y$, so $\mathrm{Ad}(k_1^t)V_X + \mathrm{Ad}(k'_0)V_Y = \mathfrak{p}$. It follows that

$$V_X + \mathrm{Ad}((k_1^t)^{-1}k'_0)V_Y = \mathfrak{p}.$$

(ii) Proof in the case $X[p;0], Y[p;0]$

This case must be treated separately because the predecessors X', Y' and consequently the sets N_X and N_Y are different than in case **(i)**. The structure of the induction proof is identical as in **(i)**, with the Steps 2 and 3 executed together.

We choose both predecessors $X'[p-1;0], Y'[p-1;0]$ and arrange X, X', Y, Y' in the same way as we did in the first part of the proof with $Y[\mathbf{t};v]$ and $Y'[\mathbf{t}';v]$. In that case, $N_X = \{X_1, Z_{12}, \ldots, Z_{1p}\} = N_Y$ and the space $\mathrm{Ad}(k'_0)(N_Y)$ is generated by

$$\left[\begin{array}{c|c} 0 & \beta_i^T \\ \hline -\alpha_i & 0 \end{array}\right]^s, \ i = 1, \ldots, p-1 \ \text{ and } \ \left[\begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array}\right]^s. \tag{12}$$

Recall that

$$Z_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array}\right]^s, \ j = 2, \ldots, p. \tag{13}$$

We want to show that the matrices in (12) together with those of (13) are linearly independent for a $k'_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ for which the equality (6) holds. Note that if $k'_0 = \begin{bmatrix} -I_{p-1} & 0 \\ 0 & I_p \end{bmatrix}$ (p odd) or $k'_0 = \begin{bmatrix} I_{p-1} & 0 \\ 0 & -I_p \end{bmatrix}$ (p even) then the matrices (12) and (13) are linearly independent. Using once more the reasonning in Lemma 11, this implies that the set of matrices k'_0 for which this is true, is open and dense in $\mathbf{SO}(p-1) \times \mathbf{SO}(p)$.

We conclude that if $N'_X = N_X \backslash \{X_1\}$ then span $(N'_X + V_{X'}) + \mathrm{Ad}(k'_0)V_Y$ has the form given in (11).

We reproduce the previous Step 4 and Step 5 using $v_1 = X_1^+$. The rest follows.

2. Proof that the case (p,q) **implies the case** $(p, q+1)$

$=====\Rightarrow$We will show by induction that for any $q > p$, there exists a matrix $k \in K$ such that (5) holds.$\Leftarrow=====$We know by the first part of the proof that this is true for $\mathbf{SO}_0(p, p+1)$.

Assume that X and Y are eligible in $\mathbf{SO}_0(p, q+1)$. Their configurations are eligible in $\mathbf{SO}_0(p, q)$. We write X', Y' when we work in $\mathbf{SO}_0(p, q)$.

We embed $K' = \mathbf{SO}(p) \times \mathbf{SO}(q)$ in $K = \mathbf{SO}(p) \times \mathbf{SO}(q+1)$ in the following way

$$K' = \begin{bmatrix} \mathbf{SO}(p) & & \\ & \mathbf{SO}(q) & \\ & & 1 \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(q+1) \end{bmatrix}.$$

The space $\mathfrak{p}'$ is formed by matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B are $p \times q$ matrices. We embed $\mathfrak{p}'$ in $\mathfrak{p}$ by adding a last column of zeros to B.

Step 1. We suppose that there exists a matrix $k_0 \in K'$ such that

$$V_{X'} + \mathrm{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \tag{14}$$

Then, by [7, Lemma 3.3], for any permutations s_1 and s_2 of the diagonal entries of $\mathcal{D}_{X'}$ and $\mathcal{D}_{Y'}$, there exists $k_0 \in K'$ such that

$$V_{s_1X'} + \mathrm{Ad}(k_0)V_{s_2Y'} = \mathfrak{p}'$$

so we can permute the elements of X' and Y' in a convenient way and still have the equality (14). We will arrange them in the following way (where the stars denote nonzero entries):

$$\mathcal{D}_{X'} = \mathrm{diag}[\overbrace{0, \ldots, 0}^{u}, \star, \ldots, \star], \quad \mathcal{D}_{Y'} = \mathrm{diag}[\star, \ldots, \star, \overbrace{0, \ldots, 0}^{v}].$$

Let us denote $k_{01} \in \mathbf{SO}(p)$ and $k_{02} \in \mathbf{SO}(q)$ the matrices compositing k_0 corresponding in (14) to such X' and Y'. We can suppose that the matrix k_{01} is total.

By the eligibility of X and Y, $u + v \le p$, so no two zeros in $\mathcal{D}_{X'}$ and $\mathcal{D}_{Y'}$ are at the same position.

Let $N = \{X_{i,q+1}\}_{i=1}^{p}$. We set $N_X := V_X \cap N = \{X_{u+1,q+1}, \ldots, X_{p,q+1}\}, N_Y := V_Y \cap N = \{X_{1,q+1}, \ldots, X_{p-v,q+1}\}$. We have $p - v \ge u$.

Step 2. Let

$$k_1 = \begin{bmatrix} k_{01} & & \\ & k_{02} & \\ & & 1 \end{bmatrix}$$

where $k_{01} \in \mathbf{SO}(p)$ and $k_{02} \in \mathbf{SO}(q)$ are the blocks compositing k_0. We then have

$$V_{X'} + \mathrm{Ad}(k_1)V_{Y'} = \begin{bmatrix} \mathfrak{p}' & \mathbf{0} \end{bmatrix}^s. \tag{15}$$

The space $V_X + \mathrm{Ad}(k_1)V_Y$ contains, in addition to the matrices in (15), the linear span of $N_X + \mathrm{Ad}(k_1)N_Y$.

Denote the columns of the matrix k_{01} by $\mathbf{c}_1, \ldots, \mathbf{c}_p$. By block multiplication in $\mathbf{SO}_0(p, q+1)$, we obtain

$$\mathrm{Ad}(k_1) X_{j,q+1} = \begin{bmatrix} 0_{p\times q} & \mathbf{c}_j \end{bmatrix}^s .$$

This implies that the linear span of $N_X + \mathrm{Ad}(k_1) N_Y$ contains the following symmetric matrices:

$$\begin{bmatrix} 0_{p\times q} & \mathbf{c}_1 \end{bmatrix}^s, \ldots, \begin{bmatrix} 0_{p\times q} & \mathbf{c}_u \end{bmatrix}^s, \begin{bmatrix} 0_{p\times q} & \mathbf{e}_{u+1} \end{bmatrix}^s, \ldots, \begin{bmatrix} 0_{p\times q} & \mathbf{e}_p \end{bmatrix}^s,$$

which are linearly independent by the totality of k_{01}. Thus $V_X + \mathrm{Ad}(k_1) V_Y = \mathfrak{p}$. ∎

$=====\Rightarrow$

We conclude this section with an example to illustrate our proof.

Example 14 *Consider* $X = X[2;1], Y = Y[1,1;1]$ *in* $\mathfrak{so}(3,4)$*. We write* X *and* Y *in such a way that* $\mathcal{D}_X = \mathrm{diag}[0, a, a]$ *and* $\mathcal{D}_Y = \mathrm{diag}[b, c, 0]$*. Their predecessors in* $\mathfrak{so}(2,3)$ *are* X' *and* Y' *such that* $\mathcal{D}_{X'} = \mathrm{diag}[a, a]$ *and* $\mathcal{D}_{Y'} = \mathrm{diag}[c, 0]$*.*

Note that X *and* Y *form an eligible pair and so are* $X' = X[2;0]$ *and* $Y' = Y'[1;1]$*. In Step 1, we show that there exists a matrix* $k_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_{0,1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_{02} \end{bmatrix}$ *with* $k_{01} \in \mathbf{SO}(2)$ *and* $k_{0,2} \in \mathbf{SO}(3)$ *such that*

$$V_{X'} + Ad(k_0)\, V_{Y'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}^s$$

where $*$ *designates an arbitrary element. We have*

$$N_X = \{Z_{12}, Y_{12}, Z_{13}, Y_{13}\}, \qquad N_Y = \{X_1, Z_{12}, Y_{12}, Z_{13}, Y_{13}\}.$$

In Step 2, we observe that

$$Ad(k_0)\mathrm{span}(N_Y) = \left\{ \left[\begin{array}{c|ccc} 0 & a_1 & a_2 & a_3 \\ \hline a_4 & & & \\ a_5 & & 0 & \end{array}\right]^s, \quad a_1, \ldots a_6 \in \mathbf{R} \right\}$$

since the matrices

$$\left[\begin{array}{c|c} 0 & \beta_1^T \\ \hline -\alpha_1 & 0 \end{array}\right]^s, \left[\begin{array}{c|c} 0 & \beta_1^T \\ \hline \alpha_1 & 0 \end{array}\right]^s, \left[\begin{array}{c|c} 0 & \beta_2^T \\ \hline -\alpha_2 & 0 \end{array}\right]^s, \left[\begin{array}{c|c} 0 & \beta_2^T \\ \hline \alpha_2 & 0 \end{array}\right]^s, \left[\begin{array}{c|c} 0 & \beta_3^T \\ \hline 0 & 0 \end{array}\right]^s$$

are linearly independent. Note that in this case, there are no σ_i *and no* τ_i*.*

Now, $V_X = span \overbrace{\{Z_{12}, Y_{12}, Z_{13}, Y_{13}\}}^{N_X} \cup V_{X'}$ *while* $V_Y = span \overbrace{\{X_1, Z_{12}, Z_{13}, Z_{14}, Y_{13}\}}^{N_Y} \cup V_{Y'}$*. We can show that*

$$Ad(e^{t\,(Z_{1,2}^{+} + \theta\, Z_{1,2}^{+})})(V_{X'}) + Ad(k_0)\,(span\, N_Y \cup V_{Y'})$$
$$= \begin{bmatrix} 0 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s$$

for t small enough (with t small enough, the dimension will not decrease). Now,

$$Ad(e^{t\,(Z^+_{1,2}+\theta\,Z^+_{1,2})})\,\overbrace{(span\{Z_{12}\}\cup V_{X'})}^{\subset V_X}+Ad(k_0)\,\overbrace{(span\,N_Y\cup V_{Y'})}^{V_Y}=\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s=\mathfrak{p}$$

if t close to 0 since $Ad(e^{t\,(Z^+_{12}+\theta\,Z^+_{12})})(Z_{12})=\cos(4\,t)\,Z_{12}+2\,\sin(4\,t)\,(A_1+A_2)$. Therefore,

$$V_X+Ad(\overbrace{e^{-t\,(Z^+_{1,2}+\theta\,Z^+_{1,2})}\,k_0}^{k})\,V_Y=Ad(e^{-t\,(Z^+_{1,2}+\theta\,Z^+_{1,2})})\mathfrak{p}=\mathfrak{p}$$

which means that the density exists.

⇐====

5 Applications

====⇒We now extend our results to the complex and quaternion cases.⇐====

Recall that $\mathbf{SU}(p,q)$ is the subgroup of $\mathbf{SL}(p+q,\mathbf{C})$ such that $g^*\,I_{p,q}\,g=I_{p,q}$ while $\mathbf{Sp}(p,q)$ is the subgroup of $\mathbf{SL}(p+q,\mathbf{H})$ such that $g^*\,I_{p,q}\,g=I_{p,q}$. Their respective maximal compact subgroups are $\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$ and $\mathbf{Sp}(p)\times\mathbf{Sp}(q)\equiv\mathbf{SU}(p,\mathbf{H})\times\mathbf{SU}(q,\mathbf{H})$.

Their subspaces $\mathfrak{p}$ can be described as $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ where B is an arbitrary complex (respectively quaternionic) matrix of size $p\times q$. The Cartan subalgebra $\mathfrak{a}$ is chosen in the same way as for $\mathfrak{so}(p,q)$.

Corollary 15 *Consider the symmetric spaces $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$, $q>p$.*

Let X, $Y\in\mathfrak{a}$. Then the measure $\delta^\natural_{e^X}\star\delta^\natural_{e^Y}$ is absolutely continuous if and only if X and Y are eligible, as defined in Definition 3.

Proof: Let X, $Y\in\mathfrak{a}$. If they are eligible then since

$$a(e^X\,(\mathbf{SO}(p)\times\mathbf{SO}(q))\,e^Y)\subset a(e^X\,\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))\,e^Y)\subset a(e^X\,(\mathbf{Sp}(p)\times\mathbf{Sp}(q))\,e^Y),$$

it follows from Theorem 13 that these sets have nonempty interior. Hence the density exists in all three cases.

On the other hand, given Lemma 4, one can reproduce Proposition 5 using $\mathbf{F}=\mathbf{C}$ and $\mathbf{F}=\mathbf{H}$ to show that the eligibility condition is necessary in the complex and quaternionic cases. ∎

We will conclude this paper with two further applications.

Proposition 16 *Let X and $Y\in\mathfrak{a}$ be such that $\left(\delta^\natural_{e^X}\right)^{*2}$ and $\left(\delta^\natural_{e^Y}\right)^{*2}$ are absolutely continuous. Then $\delta^\natural_{e^X}*\delta^\natural_{e^Y}$ is absolutely continuous.*

Proof: =====⇒Let $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$.⇐=====We know that the couple (X, X) is eligible; therefore

$$2\,\max\{\mathbf{s}, 2\,u\} \leq 2\,p.$$

In the same manner, $\max\{\mathbf{t}, 2\,v\} \leq p$. Hence,

$$\max\{\mathbf{s}, 2\,u\} + \max\{\mathbf{t}, 2\,v\} \leq p + p = 2\,p$$

which means that X and Y are eligible. Consequently, $\delta^\natural_{e^X} * \delta^\natural_{e^Y}$ is absolutely continuous. ∎

If $X \in \mathfrak{a}$ and $X \neq 0$, it is important to know for which convolution powers l the measure $\left(\delta^\natural_{e^X}\right)^l$ is absolutely continuous. This problem is equivalent to the study of the absolute continuity of convolution powers of uniform orbital measures $\delta^\natural_g = m_K * \delta_g * m_K$ for $g \notin K$.

It was proved in [8, Corollary 7] that it is always the case for $l \geq r + 1$, where r is the rank of the symmetric space G/K. It was also conjectured ([8, Conjecture 10]) that $r + 1$ is optimal for this property, which was effectively proved for symmetric spaces of type A_n([8, Corollary 18]). In the following theorem, the conjecture is shown not to hold on symmetric spaces of type =====⇒B_p⇐=====, where $r = p$. Thanks to the rich structure of the root system B_p, already all p-th powers of orbital measures are absolutely continuous and p is optimal for this property.

Theorem 17 *On symmetric spaces* $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ *and* $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, $q > p$, *for every nonzero* $X \in \mathfrak{a}$, *the measure* $(\delta^\natural_{e^X})^p$ *is absolutely continuous. Moreover,* p *is the smallest value for which this is true: if* X *has a configuration* $[1; p-1]$ *then the measure* $(\delta^\natural_{e^X})^{p-1}$ *is singular.*

Proof: We will write S^l_X for the set $a(e^X\,K\,\ldots, K\,e^X)$ where the factor e^X appears l times. Note that $(\delta^\natural_{e^X})^l$ is absolutely continuous if and only if S^l_X has nonempty interior.

We prove first that for $l < p$, the measure $(\delta^\natural_{e^X})^l$ may not be absolutely continuous. Let $X = X[1; p-1]$. Using Lemma 4 repeatedly, as in the proof of Proposition 5, we show that for $l < p$, there are at least $p - l$ diagonal entries of $\mathcal{D}_H$ which are equal to 0 for every $H \in S^l_X$. Consequently, S^l_X has empty interior and $(\delta^\natural_{e^X})^l$ is not absolutely continuous when $l \leq p - 1$.

We will now show that $(\delta^\natural_{e^X})^p$ has a density for every $X \neq 0$.

If $X = X[\mathbf{s}; 0]$ then the measure $(\delta^\natural_{e^X})^2$ is already absolutely continuous (the couple (X, X) is eligible). Suppose then that $X = X[\mathbf{s}; u] \in \overline{\mathfrak{a}^+}$, $u > 0$.

Remark that if $H \in S^l_X$ then $a(e^X\,K\,e^H) \subset S^{l+1}_X$. Indeed, we have $e^X\,k_1\,E^X\,\ldots\,k_{l-1}\,e^X = k_a\,e^H\,k_b$ and therefore $a(e^X\,K\,e^H) = a(e^X\,K\,k_a\,e^H\,k_b) = a(e^X\,K\,e^X\,k_1\,\ldots\,k_{l-1}\,e^X) \subset S^{l+1}_X$.

We claim that there exists $H \in S^{p-1}_X$ such that $H = H[1^{p-1}; 1]$ or $H \in \mathfrak{a}^+$.

We prove the claim using induction on p. If $p = 2$ then $S^{p-1}_X = \{X\}$ and the result follows (in that case, u cannot be higher than 1 for $X \neq 0$).

Suppose that the claim is true for $p - 1 \geq 2$. Let $K_0 = \begin{bmatrix} \mathbf{SO}(p-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{SO}(q-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Consider the set $B = a(e^X\,K_0\,e^X\,\ldots\,e^X)$ with $p - 1$ factors e^X. By the induction hypothesis, there exists $H_0 \in B$ with $H_0 = H_0[1^{p-2}; 2]$ or $H_0 = H_0[1^{p-1}; 1]$. In the second case, we are done.

If $H_0 = H_0[1^{p-2}; 2] \in B$, we can assume that the diagonal entries which are 0 in $\mathcal{D}_X$ and in H_0 are at the end. We note that X and H_0 considered without their last entries are eligible in $\mathbf{SO}_0(p-1, q-1)$, their configurations being $[\mathbf{s}; u-1]$ and $[1^{p-2}; 1]$ respectively. Hence $a(e^X\, K_0\, e^{H_0})$ has nonempty interior in the subspace $\overline{\mathfrak{a}^+} \cap \{H_p = 0\}$. Therefore, there exists $H \in a(e^X\, K_0\, e^{H_0}) \subset S_X^{p-1}$ with $H = H[1^{p-1}; 1]$ which proves the claim.

To conclude, we take $H \in S_X^{p-1}$ with $H \in \mathfrak{a}^+$ or $H = H[1^{p-1}; 1]$. In both cases, X and H are eligible, so by Corollary 15 the set $a(e^X\, K\, e^H)$ has nonempty interior. As $a(e^X\, K\, e^H) \subset S_X^p$, this ends the proof. ∎

6 Conclusion

With this paper and with [7], we have now obtained sharp criteria on singular X and Y for the existence of the density of $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ for the root systems of type A_n and type B_p. Thanks to [8] and Theorem 17 of the present paper, sharp criteria are now given for the l-th convolution powers $(\delta^\natural_{e^X})^l$ to be absolutely continuous for any $X \neq 0$, $X \in \mathfrak{a}$.

Although there is considerable similarity between the criteria for both type of spaces, a characterization of eligibility that would be applicable for all Riemannian symmetric spaces of non-compact type has yet to emerge. The solution of the second problem in Theorem 17 seems to indicate that the answer may depend on the type of the symmetric space.

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